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Contiguous Relations for the Double and Multiple Half-Range Fourier Series and I-Function

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Abstract

The aim of this paper is to study a natural generalization of the well-known, interesting and useful Fox H-function in to generalized function of single variable I-function and Half-Range Fourier series. In this paper, we have presented four double integrals of I-function and utilize them to obtain four double half range Fourier series of the I-function analogues to our one double integral and one double and one double half-range Fourier series of I-function respectively. The I-function and the results derived presented in this paper are basic in nature and may include a number of known and new results.

1 INTRODUCTION

Almost all research papers on Fourier series of generalized hypergeometric function have been discussed and listed in [4], [5] and [9]. It is important to note that all the Fourier series mentioned above are half-range Fourier series. This paper appears to be an attempt in the field of double and multiple Fourier series of the generalized hypergeometric functions. Parashar [12], Anandani [6], Saxena [11] established some Fourier series of Fox's H-function [10].

The subject of Fourier series of the generalized hypergeometric function play an important role in the development of theory of special functions and certain Fourier series of the generalized hypergeometric function enable us to obtain general solutions of some boundary value problems.

The I-function, which is more general than the Fox's H-function [10], defined by V.P. Saxena [2], by means of the following Mellin-Barnes type contour integral:

$$I[z]=I^{m,n}_{p_i,q_i:r}[z]=I^{m,n}_{p_i,q_i:r}\left[z\,\middle|\begin{matrix}(a_j,\alpha_j)_{1,n};(a_{ji},\alpha_{ji})_{n+1,p_i}\\(b_j,\beta_j)_{1,m};(b_{ji},\beta_{ji})_{m+1,q_i}\end{matrix}\right]=\frac{1}{2\pi i}\int_L \phi(\xi)z^{\xi}d\xi \tag{1.1}$$

where

$$\phi(\xi)=\frac{\prod_{j=1}^{m}\Gamma(b_j-\beta_j\xi)\prod_{j=1}^{n}\Gamma(1-a_j+\alpha_j\xi)}{\sum_{i=1}^{r}\left\{\prod_{j=m+1}^{q_i}\Gamma(1-b_{ji}+\beta_{ji}\xi)\prod_{j=n+1}^{p_i}\Gamma(a_{ji}-\alpha_{ji}\xi)\right\}} \tag{1.2}$$

p_i, q_i $(i = 1,\ldots,r)$, m, n are integers satisfying $0\le n\le p_i, 0\le m\le q_i; \alpha_i,\beta_i,\alpha_{ji},\beta_{ji}$ are real and positive and a_j, b_j, a_{ji}, b_{ji} are complex numbers. L is suitable contour of the Mellin-Barnes type running from $\gamma-i\alpha$ to $\gamma+i\alpha$ (γ is real) in the

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complex ξ–plane. Detail regarding existence conditions and various parametric restrictions of I-function, we may refer [2].

Remark: For r = 1, (1.1) reduces to the Fox's H-function [2]:

$$I^{m,n}_{p_i,q_i:1}\left[z\left|\begin{matrix}(a_j,\alpha_j)_{1,n};(a_{ji},\alpha_{ji})_{n+1,p_i}\\(b_j,\beta_j)_{1,m};(b_{ji},\beta_{ji})_{m+1,q_i}\end{matrix}\right.\right] = H^{m,n}_{p,q}\left[z\left|\begin{matrix}(a_j,\alpha_j)_{1,n};(a_j,\alpha_j)_{n+1,p}\\(b_j,\beta_j)_{1,m};(b_j,\beta_j)_{m+1,q}\end{matrix}\right.\right]$$

The following are the double orthogonality properties of sine and cosine function, which, may be verified easily:

$$\int_0^\pi\int_0^\pi \sin mx \sin rx \sin ny \sin ty \, dx\, dy = \begin{cases}\pi^2/4, m=r, n=t\\ 0 \quad , m\neq r \text{ or } n\neq t\end{cases} \tag{1.3}$$

$$\int_0^\pi\int_0^\pi \sin mx \sin rx \cos ny \cos ty \, dx\, dy = \begin{cases}\pi^2/4, m=r, n=t\\ 0 \quad , m\neq r \text{ or } n\neq t\\ \pi^2/2, m=r, n=t=0\end{cases} \tag{1.4}$$

$$\int_0^\pi\int_0^\pi \sin mx \sin rx \cos ny \cos ty \, dx\, dy = \begin{cases}\pi^2/4, m=r, n=t\\ 0 \quad , m\neq r \text{ or } n\neq t\\ \pi^2/2, m=r=0, n=t\end{cases} \tag{1.5}$$

$$\int_0^\pi\int_0^\pi \cos mx \cos rx \cos ny \cos ty \, dx\, dy = \begin{cases}\pi^2/4, m=r, n=t\\ 0 \quad , m\neq r \text{ or } n\neq t\\ \pi^2/2, m=r=0, n=t\\ \pi^2, \quad m=r=n=t=0\end{cases} \tag{1.6}$$

The following integrals [7, p.372, (1) and (8)]

$$\int_0^\pi (\sin x)^{\mu-1}\sin \lambda x\, dx = \frac{\pi \sin\frac{\lambda\pi}{2}\Gamma\mu}{2^{\mu-1}\Gamma\left(\frac{\mu+\lambda+1}{2}\right)\Gamma\left(\frac{\mu-\lambda+1}{2}\right)}, \text{ Re } \mu > 0 \tag{1.7}$$

$$\int_0^\pi (\sin x)^{\mu-1}\cos \lambda x\, dx = \frac{\pi \cos\frac{\lambda\pi}{2}\Gamma\mu}{2^{\mu-1}\Gamma\left(\frac{\mu+\lambda+1}{2}\right)\Gamma\left(\frac{\mu-\lambda+1}{2}\right)}, \text{ Re } \mu > 0 \tag{1.8}$$

The sake of brevity K_1 and K_2 are positive integers, the symbol $\Omega(K_1,\mu)$ represents the set of parameters $\frac{\mu}{K_1},\frac{\mu+1}{K_1},...,\frac{\mu+K_1-1}{K_1}$ and the expression $\Omega\left(K,\frac{1-\mu\pm\lambda}{2}\right)$ stands for $\Omega\left(K_1,\frac{1-\mu+\lambda}{2}\right),\Omega\left(K_1\frac{1-\mu-\lambda}{2}\right)$.

The multiplication formula for the Gamma-function [1, p. 4(11)] is given by

$$\Gamma(WZ)=(2\pi)^{\frac{1}{2}-\frac{W}{2}}W^{(WZ)-1/2}\prod_{r=0}^{W-1}\Gamma\left(Z+\frac{r}{m}\right) \tag{1.9}$$

where W is a positive integer.

2 Main Results

In this section, we have evaluated certain double integrals involving the product of the I-function with multiplication for Gamma-function.

First Integral

$$I_1 = \int_0^{\pi}\int_0^{\pi} (\sin x)^{\rho-1} (\sin y)^{\sigma-1} \sin ux \sin vy \; I(x,y)\, dx\, dy$$

$$= \frac{\pi \sin\frac{u\pi}{2} \sin\frac{v\pi}{2}}{\sqrt{K_1K_2}} I^{m,n+2K_1+2K_2}_{p_i+2K_1+2K_2,\, q_i+2K_1+2K_2 : r}\left[z \left| \begin{matrix} \Omega(2K_1,1-\rho),\Omega(2K_2,1-\sigma),(a_j,\alpha_j)_{1,n};(a_{ji},\alpha_{ji})_{n+1,p_i} \\ (b_j,\beta_j)_{1,m};(b_{ji},\beta_{ji})_{m+1,q_i},\Omega\left(K_1,\frac{1+\rho\pm u}{2}\right),\Omega\left(K_2,\frac{1+\sigma\pm v}{2}\right) \end{matrix} \right. \right] \tag{2.1}$$

where

$$I(x,y) = I^{m,n}_{p_i,q_i:r}\left[z\,(\sin x)^{2K_1} (\sin y)^{2K_2} \left| \begin{matrix} (a_j,\alpha_j)_{1,n};(a_{ji},\alpha_{ji})_{n+1,p_i} \\ (b_j,\beta_j)_{1,m};(b_{ji},\beta_{ji})_{m+1,q_i} \end{matrix} \right. \right]$$

and $2(m+n) > p+q, |\arg z| < \left(m+n-\frac{p}{2}-\frac{q}{2}\right)\pi, \; \operatorname{Re}(\rho+2K_1b_j)>0, \operatorname{Re}(\sigma+2K_2b_j)>0, \; j=1,...,m$.

Proof. In order to prove (2.1), expressing the I-function in the integrand as the Mellin-Barnes type contour integral (1.1) and interchanging the order of integrations, we have

$$\int_0^{\pi}\int_0^{\pi} (\sin x)^{\rho-1} (\sin y)^{\sigma-1} \sin ux \sin vy \times \frac{1}{2\pi i}\int_L \phi(\xi)[z(\sin x)^{2K_1} (\sin y)^{2K_2}]^{\xi} d\xi\, dx\, dy$$

$$= \frac{1}{2\pi i}\int_L \phi(\xi) z^{\xi}\left[\int_0^{\pi} (\sin x)^{\rho+2\xi K_1-1} \sin ux\, dx \int_0^{\pi} (\sin y)^{\sigma+2\xi K_2-1} \sin vy \right] d\xi$$

Evaluating the inner-integrals with the help of (1.7) and using the multiplication formula for Gamma function (1.9) and on applying (1.2), we get

$$= \frac{\pi \sin\frac{u\pi}{2} \sin\frac{v\pi}{2}}{\sqrt{K_1}\sqrt{K_2}} \frac{1}{2\pi i}\int_L \frac{\prod_{j=1}^{m}\Gamma(b_j-\beta_j\xi)\prod_{j=1}^{n}\Gamma(1-a_j+\alpha_j\xi)}{\sum_{r=1}^{r}\left\{\prod_{j=m+1}^{q_i}\Gamma(1-b_{ji}+\beta_{ji}\xi)\prod_{j=n+1}^{p_i}\Gamma(a_{ji}-\alpha_{ji}\xi)\right\}}$$

$$\times \frac{\prod_{t=0}^{2K_1-1}\Gamma\left(\frac{\rho+t}{2k_1}\right)+\xi \prod_{t=0}^{2K_2-1}\Gamma\left(\frac{\sigma+t}{2K_2}\right)+\xi}{\prod_{t=0}^{K_1-1}\Gamma\frac{((\rho+u+1)/2)+t}{K_1}+\xi\prod_{t=0}^{K_1-1}\Gamma\frac{((\rho-u+1)/2)+t}{K_1}+\xi\prod_{t=0}^{K_2-1}\Gamma\frac{((\rho+v+1)/2)+t}{K_2}+\xi} \; \frac{1}{\prod_{t=0}^{K_2-1}\Gamma\frac{((\sigma-v+1)/2)+t}{K_2}+\xi} z^{\xi} d\xi$$

After little arrangement, we finally arrive at (2.1).

Second Integral

$$I_2 = \int_0^{\pi}\int_0^{\pi} (\sin x)^{\rho-1} (\sin y)^{\sigma-1} \sin ux \cos vy \; I(x,y)\, dx\, dy$$

$$= \frac{\pi \sin\frac{u\pi}{2} \cos\frac{v\pi}{2}}{\sqrt{K_1K_2}} I^{m,n+2K_1+2K_2}_{p_i+2K_1+2K_2,\, q_i+2K_1+2K_2 : r}\left[z \left| \begin{matrix} \Omega(2K_1,1-\rho),\Omega(2K_2,1-\sigma),(a_j,\alpha_j)_{1,n};(a_{ji},\alpha_{ji})_{n+1,p_i} \\ (b_j,\beta_j)_{1,m};(b_{ji},\beta_{ji})_{m+1,q_i},\Omega\left(K_1,\frac{1+\rho\pm u}{2}\right),\Omega\left(K_2,\frac{1+\sigma\pm v}{2}\right) \end{matrix} \right. \right] \tag{2.2}$$

provided $2(m+n) > p+q, |\arg z| < \left(m+n-\frac{p}{2}-\frac{q}{2}\right)\pi, \; \operatorname{Re}(\rho+2K_1b_j)>0, \operatorname{Re}(\sigma+2K_2b_j)>0, \; j=1,2,...,m$.

Proof. On applying the same procedure as result (2.1) as given in section 2, the integral (2.2) is established.

Third Integral

$$I_3 = \int_0^{\pi}\int_0^{\pi} (\sin x)^{\rho-1}(\sin y)^{\sigma-1}\cos ux \sin vy \; I(x,y)\,dx\,dy$$

$$= \frac{\pi \cos\frac{u\pi}{2}\sin\frac{v\pi}{2}}{\sqrt{K_1K_2}} I^{m,n+2K_1+2K_2}_{p_i+2K_1+2K_2,q_i+2K_1+2K_2:r}\left[z \left| \begin{matrix} \Omega(2K_1,1-\rho),\Omega(2K_2,1-\sigma),(a_j,\alpha_j)_{1,n};(a_{ji},\alpha_{ji})_{n+1,p_i} \\ (b_j,\beta_j)_{1,m};(b_{ji},\beta_{ji})_{m+1,q_i},\Omega\left(K_1,\frac{1+\rho\pm u}{2}\right),\Omega\left(K_2,\frac{1+\sigma\pm v}{2}\right) \end{matrix}\right.\right] \quad (2.3)$$

provided $2(m+n)>(p+q), |\arg z|<\left(m+n-\frac{p}{2}-\frac{q}{2}\right)\pi,\ \text{Re}(\rho+2K_1b_j)>0, \text{Re}(\sigma+2K_2b_j)>0,\ j=1,2,...,m$

Proof. On applying the similar procedure as result (2.1) as given in section 2, the integral (2.3) is established.

Fourth Integral

$$I_4 = \int_0^{\pi}\int_0^{\pi} (\sin x)^{\rho-1}(\sin y)^{\sigma-1}\cos ux \cos vy \; I(x,y)\,dx\,dy$$

$$= \frac{\pi \cos\frac{u\pi}{2}\cos\frac{v\pi}{2}}{\sqrt{K_1K_2}} I^{m,n+2K_1+2K_2}_{p_i+2K_1+2K_2,q_i+2K_1+2K_2:r}\left[z \left| \begin{matrix} \Omega(2K_1,1-\rho),\Omega(2K_2,1-\sigma),(a_j,\alpha_j)_{1,n};(a_{ji},\alpha_{ji})_{n+1,p_i} \\ (b_j,\beta_j)_{1,m};(b_{ji},\beta_{ji})_{m+1,q_i},\Omega\left(K_1,\frac{1+\rho\pm u}{2}\right),\Omega\left(K_2,\frac{1+\sigma\pm v}{2}\right) \end{matrix}\right.\right] \quad (2.4)$$

provided $2(m+n)>(p+q), |\arg z|<\left(m+n-\frac{p}{2}-\frac{q}{2}\right)\pi,\ \text{Re}(\rho+2K_1b_j)>0, \text{Re}(\sigma+2K_2b_j)>0,\ j=1,2,...,m.$

Proof. On applying the similar procedure as result (2.1) as given in section 2, the result (2.4) is obtained.

Fifth Integral

The following multiple integral leads to the known result (2.1) can be established on applying the procedure as given in section 2 with help of (1.7):

$$I_5 = \int_0^{\pi}\int_0^{\pi}...\int_0^{\pi} (\sin x_1)^{\rho_1-1}(\sin x_2)^{\rho_2-1}...(\sin x_n)^{\rho_n-1} \sin u_1x_1 \sin u_2x_2 ... \sin u_nx_n \; I(x_1,x_2,...,x_n)\,dx_1\,dx_2...dx_n$$

$$= \frac{(\pi)^{n/2}\sin\frac{u_1\pi}{2}\sin\frac{u_2\pi}{2}...\sin\frac{u_n\pi}{2}}{\sqrt{K_1K_2...K_n}} I^{m,n+2K_1+2K_2+...+2K_n}_{p_i+2K_1+2K_2+...+2K_n,q_i+2K_1+2K_2+...+2K_n:r}\left[z \left| \begin{matrix} \Omega(2K_1,1-\rho_1),...,\Omega(2K_n,1-\rho_n),(a_j,\alpha_j)_{1,n};(a_{ji},\alpha_{ji})_{n+1,p_i} \\ (b_j,\beta_j)_{1,m};(b_{ji},\beta_{ji})_{m+1,q_i},\Omega\left(K_1,\frac{1+\rho_1\pm u}{2}\right),...,\Omega\left(K_n,\frac{1+\rho_n\pm u}{2}\right) \end{matrix}\right.\right]$$

(2.5)

provided $2(m+n)>p+q, |\arg z|<\left(m+n-\frac{p}{2}-\frac{q}{2}\right)\pi,\ \text{Re}(\rho+2K_1b_j)>0, \text{Re}(\sigma+2K_2b_j)>0,\ j=1,2,...,m.$

where $I(x,y) = I^{m,n}_{p_i,q_i:r}\left[z\,(\sin x)^{2K_1}(\sin y)^{2K_2}...(\sin x)^{2k_n} \left| \begin{matrix} (a_j,\alpha_j)_{1,n};(a_{ji},\alpha_{ji})_{n+1,p_i} \\ (b_j,\beta_j)_{1,m};(b_{ji},\beta_{ji})_{m+1,q_i} \end{matrix}\right.\right].$

The multiple integrals analogous to (2.2), (2.3), (2.3) can also be established by applying the similar procedure as given in section 2.

3 THE DOUBLE HALF-RANGE FOURIER SERIES

In this section we derive the double half-range Fourier series with the help of (2.1), (2.2), (2.3), (2.4), we have following results to be established are

(A) $$\psi(x,y) = \frac{4}{\pi\sqrt{K_1K_2}}\sum_{\lambda=1}^{\infty}\sum_{\mu=1}^{\infty} \sin\frac{\lambda\pi}{2}\sin\frac{\mu\pi}{2} I(\lambda,\mu)\sin\lambda x \sin\mu y \quad (3.1)$$

(B) $$\psi(x,y)=\frac{4}{\pi\sqrt{K_1K_2}}\sum_{\lambda=1}^{\infty}\sum_{\mu=0}^{\infty}\sin\frac{\lambda\pi}{2}\cos\frac{\mu\pi}{2}I(\lambda,\mu)\sin\lambda x\cos\mu y \tag{3.2}$$

(C) $$\psi(x,y)=\frac{4}{\pi\sqrt{K_1K_2}}\sum_{\lambda=0}^{\infty}\sum_{\mu=1}^{\infty}\cos\frac{\lambda\pi}{2}\sin\frac{\mu\pi}{2}I(\lambda,\mu)\cos\lambda x\sin\mu y \tag{3.3}$$

(D) $$\psi(x,y)=\frac{4}{\pi\sqrt{K_1K_2}}\sum_{\lambda=0}^{\infty}\sum_{\mu=0}^{\infty}\cos\frac{\lambda\pi}{2}\cos\frac{\mu\pi}{2}I(\lambda,\mu)\cos\lambda x\cos\mu y \tag{3.4}$$

where $2(m+n)>p+q, |\arg z|<\left(m+n-\frac{p}{2}-\frac{q}{2}\right)\pi,\quad \mathrm{Re}(\rho+2K_1b_j)>0, \mathrm{Re}(\sigma+2K_2b_j)>0,\ j=1,2,...,m.$

and $\psi(x,y)=(\sin x)^{\rho-1}(\sin y)^{\sigma-1}I(x,y).$

Proof.

(A) Let

$$\psi(x,y)=(\sin x)^{\rho-1}(\sin y)^{\sigma-1}I(x,y)=\sum_{\lambda=1}^{\infty}\sum_{\mu=1}^{\infty}E_{\lambda,\mu}\sin\lambda x\sin\mu y \tag{3.5}$$

Here ψ(x,y) is continuous and bounded in the open interval (0,π).

Now multiplying both sides of (3.5) by sin(ux)sin(vy) and integrating from 0 to π with respect to both x and y, we obtain

$$\int_0^{\pi}\int_0^{\pi}(\sin x)^{\rho-1}(\sin y)^{\sigma-1}\sin ux\sin vy\ I(x,y)\,dx\,dy=\sum_{\lambda=1}^{\infty}\sum_{\mu=1}^{\infty}E_{\lambda,\mu}\int_0^{\pi}\int_0^{\pi}\sin\lambda x\sin ux\sin\mu y\sin vy\,dx\,dy$$

Now using (2.1), orthogonal property of sine function (1.3) as given in section 1, it follows that

$$E_{u,v}=\frac{4}{\pi\sqrt{K_1K_2}}\left(\sin\frac{u\pi}{2}\sin\frac{v\pi}{2}\right)I^{m,n+2K_1+2K_2}_{p_i+2K_1+2K_2,q_i+2K_1+2K_2:r}\left[z\left|\begin{matrix}\Omega(2K_1,1-\rho),\Omega(2K_2,1-\sigma),(a_j,\alpha_j)_{1,n};(a_{ji},\alpha_{ji})_{n+1,p_i}\\(b_j,\beta_j)_{1,m};(b_{ji},\beta_{ji})_{m+1,q_i},\Omega\left(K_1,\frac{1+\rho\pm u}{2}\right),\Omega\left(K_2,\frac{1+\sigma\pm v}{2}\right)\end{matrix}\right.\right]$$

$$=\frac{4}{\pi\sqrt{K_1K_2}}\left(\sin\frac{u\pi}{2}\sin\frac{v\pi}{2}\right)I(u,v) \tag{3.6}$$

where $$I(u,v)=I^{m,n+2K_1+2K_2}_{p_i+2K_1+2K_2,q_i+2K_1+2K_2:r}\left[z\left|\begin{matrix}\Omega(2K_1,1-\rho),\Omega(2K_2,1-\sigma),(a_j,\alpha_j)_{1,n};(a_{ji},\alpha_{ji})_{n+1,p_i}\\(b_j,\beta_j)_{1,m};(b_{ji},\beta_{ji})_{m+1,q_i},\Omega\left(K_1,\frac{1+\rho\pm u}{2}\right),\Omega\left(K_2,\frac{1+\sigma\pm v}{2}\right)\end{matrix}\right.\right] \tag{3.7}$$

Substituting the value of $E_{u,v}$ from eq. (3.6) in (3.5), the double half-range Fourier series (3.1) is established.

(B) To prove (3.2), let us assume

$$\psi(x,y)=\sum_{\lambda=1}^{\infty}\sum_{\mu=0}^{\infty}F_{\lambda,\mu}\sin\lambda x\cos\mu y \tag{3.8}$$

Multiplying both sides of (3.8) by sin(ux)cos(vy) and integrating from 0 to π with respect to both x and y, and using (2.2) and (1.4), we obtain

$$F_{u,v}=\frac{4}{\pi\sqrt{K_1K_2}}\left(\sin\frac{u\pi}{2}\cos\frac{v\pi}{2}\right)I(\lambda,\mu) \tag{3.9}$$

except that $F_{\lambda,0}$ is one-half of the above value.

From equation (3.8) and (3.9), the result (3.2) is established.

(C) Further, to prove (3.3), let us assume

$$\psi(x,y)=\sum_{\lambda=0}^{\infty}\sum_{\mu=1}^{\infty} G_{\lambda,\mu}\cos\lambda x\sin\mu y \tag{3.10}$$

Multiplying both the sides of (3.10) by cos(*ux*)sin(*vy*) and integrating from 0 to π with respect to both x and y and using results (2.3) and (1.4), we get

$$G_{u,v}=\frac{4}{\pi\sqrt{K_1K_2}}\left(\cos\frac{u\pi}{2}\sin\frac{v\pi}{2}\right)I(\lambda,\mu) \tag{3.11}$$

except that $G_{0,\mu}$ is one half of the above value.
From equation (3.10) and (3.11), the double half-range Fourier series (3.3) follows.

(D) To prove (3.4), consider

$$\psi(x,y)=\sum_{\lambda=0}^{\infty}\sum_{\mu=0}^{\infty} H_{\lambda,\mu}\cos\lambda x\cos\mu y. \tag{3.12}$$

Multiplying both sides of (3.12) by cos(ux)cos(vy) and integrating from 0 to π with respect to both x and y and using results (2.4) and (1.6), we get

$$H_{u,v}=\frac{4}{\pi\sqrt{K_1K_2}}\left(\cos\frac{u\pi}{2}\sin\frac{v\pi}{2}\right)I(\lambda,\mu) \tag{3.13}$$

except that $H_{0,\mu}$, $H_{\lambda,0}$ are one-half and $H_{0,0}$ is one quarter of the above value.
The double half range series (3.4) is obtained by using (3.12) and (3.13).

4 SPECIAL CASES

The following multiple half-range Fourier series leads to (3.1) can be derived on the following as given in section 3, using the integral (2.5) and the multiple orthogonality property of Sine functions analogous to (1.3), we have the following result

$$\psi(x_1,x_2,...,x_n)$$

$$=\frac{2^n}{(\pi)^{n/2}\sqrt{(K_1K_2...K_n)}}\sum_{\lambda_1=1}^{\infty}\sum_{\lambda_2=1}^{\infty}...\sum_{\lambda_n=1}^{\infty}\sin\frac{\lambda_1\pi}{2}\sin\frac{\lambda_2\pi}{2}...\sin\frac{\lambda_n\pi}{2}I(\lambda_1,\lambda_2,...,\lambda_n)\sin(\lambda_1x_1)\sin(\lambda_2x_2),...,\sin(\lambda_nx_n)$$

$$=\frac{2^n}{(\pi)^{n/2}\sqrt{(K_1K_2...K_n)}}\sum_{\lambda_1=1}^{\infty}\sum_{\lambda_2=1}^{\infty}...\sum_{\lambda_n=1}^{\infty}\sin\frac{\lambda_1\pi}{2}\sin\frac{\lambda_2\pi}{2}...\sin\frac{\lambda_n\pi}{2}\sin(\lambda_1x_1)\sin(\lambda_2x_2),...,\sin(\lambda_nx_n)$$

$$\times I^{m,n+2K_1+...+2K_n}_{p_i+2K_1+...+2K_n,q_i+2K_1+...+2K_n:r}\left[z\left|\begin{matrix}\Omega(2K_1,1-\rho_1),...,\Omega(2K_n,1-\rho_n),(a_j,\alpha_j)_{1,n};(a_{ji},\alpha_{ji})_{n+1,p_i}\\(b_j,\beta_j)_{1,m};(b_{ji},\beta_{ji})_{m+1,q_i},\Omega\left(K_1,\frac{1+\rho_1\pm\lambda}{2}\right),...,\Omega\left(K_2,\frac{1+\rho_n\pm\mu}{2}\right)\end{matrix}\right.\right] \tag{4.1}$$

provided $2(m+n)>p_i+q_i, |\arg z|<\left(m+n-\frac{p_i}{2}-\frac{q_i}{2}\right)\pi$, $\text{Re}(\rho+2K_1b_j)>0, \text{Re}(\sigma+2K_nb_j)>0$, $j=1,2,...,m.$

Also, $\psi(x_1,x_2,...,x_n)=(\sin x_1)^{\rho_1-1}(\sin x_2)^{\rho_2-1}..(\sin x_n)^{\rho_n-1}I(x_1,x_2,...,x_n)$

Similarly, the multiple half -range Fourier series analogue to (3.2), (3.4) and (3.4) can also be derived.

Particular Cases

(1) Putting $r=1$ and on specializing the parameters $\alpha_j=\beta_j=\alpha_{ji}=\beta_{ji}=\alpha$ in results (2.1), (2.2), (2.3), (2.4) and (2.5), we get the results obtained by Bajpai [4].

(2) For Double half range series again putting $r=1, \alpha_j=\beta_j=\alpha_{ji}=\beta_{ji}=\alpha$ in Section 2, we get the results obtained by Bajpai [4].

5 CONCLUDING REMARKS

The I-function, presented in this paper, is quite basic in nature. Therefore, on specializing the parameters of the function, we may obtain various other special functions such as Fox's H-function, Meijer's G-function, Wright's generalized Bessel function, Wright's generalized hypergeometric function, generalized hypergeometric function, exponential function, binomial function etc.

The results deduced in the present paper may provide better the double and multiple half range series and I-function of some simpler special functions. The results, so established may be found useful in the several interesting situation appearing in the literature on mathematical analysis, applied mathematics and mathematical physics.

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