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Certain Properties of Some Special Matrix Functions via Lie Algebra

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Abstract

In this paper, we establish a result concerning eigenvector for the product of two operators C and D defined on a Lie algebra of endomorphisms of a vector space. Further, A new method has been devised to define some properties viz. differential recurrence relations and differential equations of 2-variables generalized Hermite matrix polynomials and 2- variables matrix Laguerre polynomials to derive certain results involving these polynomial.

Keywords: Lie Algebra; Hermite Matrix Polynomial; Laguerre Matrix Polynomial; Matrix differential equation.
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1 Introduction

Special matrix functions seen on statistics, Lie group theory and number theory are well known (see e.g Constantine and Muirhead [5], Terras [21] and James [8]). These types of functions are also useful in many subject viz. physics, chemistry and mechanics see Keller and Wolfe [15], Morse and Fesbach [16] and Parter *et. al.* [17]. Recently, the classical orthogonal polynomials have been extended to the orthogonal matrix polynomials by Defez and Jodar [6], [7] and Jodar *et. al* [10], [11]. Motivated by their work, in this paper, we establish results for their polynomials using lie algebra approach.

Throughout the paper, we assume that A is a positive stable matrix in $\mathbb{C}^{\mathbb{N}\times\mathbb{N}}$, that is A satisfies the following condition:

$$\Re(\mu) > 0, \quad \text{for all } \mu \in \sigma(A), \tag{1.1}$$

where $\sigma(A)$ denotes the set of all the eigenvalues of A. If D_0 is the complex plane cut along the negative real axis and $\log(z)$ denotes the principle logarithm of z, then $z^{\frac{1}{2}}$ represents $\exp(\frac{1}{2}\log\ z)$. If the matrix $A \in \mathbb{C}^{\mathbb{N}\times\mathbb{N}}$ with $\sigma(A) \subset D_0$, then $A^{\frac{1}{2}} = \sqrt{A}$ denote the image of the matrix functional calculus acting on the matrix A.

The Hermite matrix polynomials (Jodar and Company [9]) $H_n(x; A)$ are defined as:

$$H_n^{\lambda}(x; A) = n! \sum_{k=0}^{\left[\frac{n}{2}\right]} \frac{(-1)^k}{(n-2k)!k!} (x\sqrt{2A})^{n-2k} \quad (n \geq 0) \tag{1.2}$$

and the following Rodrigues Formulla holds

$$H_n(x, A) = exp\left(\frac{Ax^2}{2}\right)(-1)^n\left(\frac{A}{2}\right)\left[\frac{d^n}{dx^n} exp\left(\frac{Ax^2}{2}\right)\right], \quad n \geq 0 \tag{1.3}$$

and satisfy the three terms recurrence relationship.

$$\begin{aligned} &H_n(x, A) = xI\sqrt{2A}H_{n-1} - 2(n-1) \\ &H_{n-2}(x, A), n \geq 1 \\ &H_{-1}(x, A) = 0, H_0(x, A) = 1 \end{aligned} \tag{1.4}$$

where, I is the identity matrix in $C^{R\times R}$.
by (Jodar and Company [9]) we also have the generating function

$$\exp\left(xt\sqrt{2A}-t^2I\right)=\sum_{n=0}^{\infty}H_n(x,A)\frac{t^n}{n!}. \tag{1.5}$$

The 2-variable generalized Hermite matrix polynomials $H_n^{\lambda}(x,y;A)$ (Batahan [2]) defined as:

$$H_n^{\lambda}(x,y;A)=n!\lambda^n\sum_{k=0}^{\left[\frac{n}{2}\right]}\frac{x\sqrt{(\frac{A}{2})^{n-2k}}y^k}{\lambda^k(n-2k)!k!}\quad (n\geq 0) \tag{1.6}$$

and specified by the generating function

$$\exp\left(\lambda(xt)\sqrt{\frac{A}{2}}+yt^2I\right)=\sum_{n=0}^{\infty}H_n^{\lambda}(x,y;A)\frac{t^n}{n!}. \tag{1.7}$$

and satisfy the recurrence relationship.

$$\begin{aligned}&H_0(x.y,A)=I,H_1(x,y,A)=x\sqrt{2A}\\&H_n(x,y,A)=y^{\frac{n}{2}}H_n(x/\sqrt{y},A)\\&H_n(x,1,a)=H_n(x,A)\end{aligned} \tag{1.8}$$

where $H_n(x,a)$ is defined in (1.2) The Laguerre matrix polynomials $L_n^{(A,\lambda)}(x)$ (Jodar *et. al.* [10]) are defined by

$$L_n^{(A,\lambda)}(x)=\sum_{k=0}^{n}\frac{(-1)^k\lambda^k}{k!(n-k)!}(A+I)_n[(A+I)_k]^{-1}x^k,\ \ n\geq 0 \tag{1.9}$$

where $(A)_n$ is the matrix Pochhammer symbol defined by

$$(A)_n=A(A+I)\cdots(A+(n-1)I),\ n\geq 1;\ (A)_0=I. \tag{1.10}$$

and specified by the generating function

$$(1-t)^{-(A+I)}\exp\left(\frac{-\lambda xt}{1-t}\right)=\sum_{n=0}^{\infty}L_n^{(A,\lambda)}(x)t^n,\ x,t\in\mathbb{C},\ |t|<1 \tag{1.11}$$

The 2- variable Laguerre matrix polynomials $L_n^{(A,\lambda)}(x)$ (Khan and Hasan [14]) are defined by

$$L_n^{(A,\lambda)}(x,y)=\sum_{k=0}^{n}\frac{(-1)^k\lambda^kx^ky^k}{k!(n-k)!}(A+I)_n[(A+I)_k]^{-1},\ \ n\geq 0 \tag{1.12}$$

where $(A)_n$ is the matrix Pochhammer symbol defined by

$$(A)_n=A(A+I)\cdots(A+(n-1)I),\ n\geq 1;\ (A)_0=I. \tag{1.13}$$

which can also be expressed in terms of the confluent hypergeometric function (Andrews [1]) as

$$L_n^{(A,\lambda)}(x,y)=\frac{\Gamma(A+(n+1)I)(\Gamma(A+I))^{-1}y^n}{\Gamma(n+1)}1\ F_1\left[-n;A+I;\frac{\lambda x}{y}\right] \tag{1.14}$$

and specified by the generating function

$$(1-yt)^{-A}\exp(-\lambda xt)=\sum_{n=0}^{\infty}L_n^{(A,\lambda)}(x,y)t^n,\ x,y,t\in\mathbb{C},\ |yt|<1 \tag{1.15}$$

Laguerre matrix polynomials studied by various authors. Jodar and Sastre[12] studied growth of Laguerre matrix polynomials on Bounded Intervals. Sastre *et al.* [19] studied the expansion of matrix functions in series of Laguerre matrix polynomials of a complex matrix parameter. Also they find application of Laguerre matrix polynomials to the numerical inversion of Laplace transforms of matrix functions see [20]. Sastre and Dafej [18] investigate asymptotic behavior of Laguerre matrix polynomials for large x and n.

The differential recurrence relations and differential equations for some matrix polynomials considered in this paper has been obtain using new technique discussed is the novelty, urgently and originality of this paper.

2 Main Result

Let $End\ V$ be the Lie algebra of endomorphisms of a vector space V, endowed with the Lie bracket $[\cdot,\cdot]$ defined by $[C,D] = CD - DC$, for every $C, D \in End\ V$. The main result of the paper is as follow:

Theorem 2.1 *Let $C, D \in End\ V$ be such that $[C,D]y_n = -y_n$, the sequence $(y_n)_n \subset V$ is defined as follows: $Cy_0 = 0$ and $Dy_n = -(n+1)y_{n+1}$, for every $n \geq 1$. Then $Cy_n = y_{n-1}$ and y_n is an eigenvector of eigenvalue of $-n$ for DC, for every $n \geq 1$.*

Proof: First, we show

$$Cy_n = y_{n-1}, \text{ for every } n \geq 1.$$

For $n = 1$, this equality is evident, because

$$[C,D]y_0 = -y_0,$$

$$C(Dy_0) - D(Cy_0) = -y_0,$$

also $Cy_0 = 0$ and $Dy_0 = -y_1$ and therefore,

$$Cy_1 = y_0$$

Now, suppose that $Cy_n = y_{n-1}$, then we have

$$[C,D]y_n = -y_n,$$

$$\Rightarrow C(Dy_n) - D(Cy_n) = -y_n,$$

$$\Rightarrow C((n+1)y_{n+1}) - D(n+\alpha)(y_{n-1}) = -y_n,$$

using linearity property and $Dy_{n-1} = ny_n$ we get

$$\Rightarrow (n+1)C(y_{n+1}) - (n+\alpha)ny_n = -y_n,$$

solving for $C(y_n + 1)$ we obtain,

$$\Rightarrow C(y_{n+1}) = y_n.$$

Therefore, by mathematical induction, $Cy_n = y_{n-1}$, for every $n \geq 1$. It follows immediately that $DCy_n = -ny_n$. Hence, y_n is an eigenvector of eigenvalue $-n$ for DC, for every $n \geq 1$.

3 Concrete Applications

In this Section we apply Theorem 2.1 to obtain certain properties of following matrix polynomials.

3.1 2-variable Generalized Hermite matrix polynomials

In this subsection we apply Theorem (2.1) to obtain differential recurrence relations and differential equations for 2-variable Hermite Generalized Hermite matrix polynomials.

Theorem 3.1 *The 2-variable generalized Hermite matrix polynomials $H_n^\lambda(x,y;A)$ (Khan and Raza [13]) in Eq. (1.6)* *satisfy the following differential equation*

$$\left[\frac{\lambda A - 2yI}{A^2\lambda}\right]\frac{\partial^2}{\partial x^2}f_n(x,y;A) - xI\frac{\partial}{\partial x}f_n(x,y;A) + nIf_n(x,y;A) = 0 \tag{3.1}$$

and the recurerence relation

$$\left(\lambda I - \frac{2y}{A}\right)\frac{\partial}{\partial x}f_n(x,y;A) = \lambda xAf_n(x,y;A) - (n+1)If_{n+1}(x,y;A) \tag{3.2}$$

or

$$\frac{\partial}{\partial x}f_n(x,y,A) = A\lambda f_{n-1}(x,y,A) \tag{3.3}$$

Proof: Let $V = \mathbb{C}^{N\times N}$, we define the operator $C, D \in End\ V$ as

$$C(x,y,t;A) = \frac{1}{\lambda tA}\frac{\partial u}{\partial x} \tag{3.4}$$

$$D(x,y,t;A) = \left(\lambda I - \frac{2y}{A}\right)t\frac{\partial u}{\partial x} - \lambda xtA \tag{3.5}$$

For $x, y, t \in \mathbb{C}$ and A is the matrix in $\mathbb{C}^{N\times N}$. We claim that the operators (3.4) and (3.5) obey the commutation relation
$[C, D]y_n = -y_n$
Indeed,

$$\begin{aligned}[C,D]u(x,y,t;A) &= C(Du(x,y,t;A)) - D(Cu(x,y,t;A)) \\ &= \frac{1}{\lambda tA}\frac{\partial}{\partial x}\left[\left(\lambda I - \frac{2y}{A}\right)t\frac{\partial u}{\partial x} - \lambda xtA\right] - \left[\left(\lambda I - \frac{2y}{A}\right)t\frac{\partial}{\partial x} - \lambda xtA\right]\frac{I}{\lambda tA}\frac{\partial u}{\partial x} \\ &= -u(x,y,t;A)\end{aligned} \tag{3.5}$$

i.e.

$$[C,D]u(x,y,t;A) = -u(x,y,t;A).$$

Now, if $u(x,y,t;A)$ assumes the form $y_n(x,y,t;A) = f_n(x,y;A)t^n \in \mathbb{C}^{N\times N}$, then we have

$$[C,D](f_n(x,y;A)t^n) = -f_n(x,y;A)t^n$$

and our claim is justified.
Now, the relation $Dy_n = -(n+1)y_{n+1}$ gives following differential recurrence relation on operator D

$$\left(\lambda I - \frac{2y}{A}\right)t\frac{\partial}{\partial x} - \lambda xtAf_n(x,y;A)t^n = -(n+1)f_{n+1}(x,y;A)t^{n+1}$$

$$\Rightarrow \left(\lambda I - \frac{2y}{A}\right)f_n'(x,y;A)t^{n+1} - \lambda xAf_n(x,y;A)t^{n+1} = -(n+1)f_{n+1}(x,y;A)t^{n+1}$$

or

$$\left(\lambda I - \frac{2y}{A}\right)\frac{\partial}{\partial x}f_n(x,y;A) = \lambda xAf_n(x,y;A) - (n+1)If_{n+1}(x,y;A) \tag{3.6}$$

And from the relation $Cy_n = y_{n-1}$ we obtain the following differential recurrence relation on operator A

$$\left[\frac{1}{\lambda tA}\frac{\partial}{\partial x}\right]f_n(x,y;A)t^n = f_{n-1}(x,y;A)t^{n-1}$$

$$\Rightarrow \frac{1}{A\lambda}f_n'(x,y;A)t^{n-1} = f_{n-1}(x,y;A)t^{n-1}$$

or

$$\frac{\partial}{\partial x}f_n(x,y,A) = A\lambda f_{n-1}(x,y,A) \tag{3.7}$$

Finally the relation $DCy_n = -ny_n$ gives

$$\left[\left(\lambda I - \frac{2y}{A}\right) t\frac{\partial}{\partial x} - \lambda xtA\right]\left[\frac{1}{\lambda tA}\frac{\partial}{\partial x}\right] f_n(x,y;A)t^n = -nf_n(x,y;A)t^n$$

$$\Rightarrow \left(\lambda I - \frac{2y}{A}\right)\frac{1}{\lambda A}\frac{\partial^2}{\partial x^2} f_n(x,y;A)t^n - xI\frac{\partial}{\partial x} f_n(x,y;A)t^n = -nIf_n(x,y;A)t^n$$

$$\Rightarrow \left[\frac{\lambda A - 2yI}{A^2\lambda}\right]\frac{\partial^2}{\partial x^2} f_n(x,y;A) - xI\frac{\partial}{\partial x} f_n(x,y;A) + nIf_n(x,y;A) = 0 \tag{3.8}$$

Now, we observe that 2-variable Hermite Generalized Hermite matrix polynomials $H_n^\lambda(x,y;A)$ is the solution of the differential equation (3.8). Further we note that the relation (3.6) and (3.7) are differential recurrence relation satisfied by 2-variable Hermite Generalized Hermite matrix polynomials $H_n^\lambda(x,y;A)$.

3.2 2- variables Laguerre matrix polynomials

In this subsection we apply Theorem (2.1) to obtain differential recurrence relations and differential equations for 2-Variables Laguerre Matrix Polynomials.

Theorem 3.2 *The 2- variable Laguerre matrix polynomials $L_n^{(A,\lambda)}(x)$ (Khan and Hasan [14]) in Eq. (1.12) satisfy the following differential equation*

$$-\frac{xy}{\lambda}\frac{\partial^2}{\partial x^2} - \left[\frac{Ay}{\lambda} + \left\{\frac{y}{\lambda}(n+1) - x\right\} I\frac{\partial}{\partial x} - A\right] f_n(x,y;A) = -nf_n(x,y;A) \tag{3.9}$$

and the recurrence relation

$$\left[\frac{x}{\lambda y}\frac{\partial}{\partial x} + \frac{n+AI}{\lambda y}\right] f_n(x,y;A) = -f_{n-1}(x,y;A) \tag{3.10}$$

or

$$\left[-y^2 I\frac{\partial}{\partial x} + I\lambda y\right] f_n(x,y;A) = -(n+1)f_{n+1}(x,y;A) \tag{3.11}$$

Proof: Let $V = \mathbb{C}^{N\times N}$, we define the operator $C, D \in End\ V$ as

$$C = -\left[\frac{x}{\lambda ty} I\frac{\partial u}{\partial x} + \frac{I}{\lambda y}\frac{\partial u}{\partial t} + \frac{1}{\lambda ty}A\right] \tag{3.12}$$

$$D = -Iy^2 t\frac{\partial u}{\partial x} + I\lambda ty \tag{3.13}$$

For $x, y, t \in \mathbb{C}$, A is the matrix in $\mathbb{C}^{N\times N}$ and I is the unit matrix in $\mathbb{C}^{N\times N}$.
We claim that the operators (3.12) and (3.13) obey the commutation relation
$[C,D]y_n = -y_n$
Indeed,

$$\begin{aligned}[C,D]u(x,y,t;A) =& C(Du(x,y,t;A)) - D(Cu(x,y,t;A)) \\ =& \left[\frac{Ix}{\lambda ty}\frac{\partial}{\partial x} - \frac{I}{\lambda y}\frac{\partial}{\partial t} - \frac{1}{\lambda ty}A\right]\left[-y^2 tI\frac{\partial u}{\partial x} + I\lambda ty\right] - \left[-y^2 tI\frac{\partial}{\partial x} + I\lambda ty\right] - \left[\frac{Ix}{\lambda ty}\frac{\partial u}{\partial x} + \frac{I}{\lambda y}\frac{\partial u}{\partial t} + \frac{I}{\lambda ty}A\right] \\ =& -u(x,y,t;A)\end{aligned} \tag{3.13}$$

i.e.

$$[C,D]u(x,y,t;A) = -u(x,y,t;A)$$

Now, if $u(x,y,t;A)$ assumes the form $u(x,y,t;A) = f_n(x,y;A)t^n \in \mathbb{C}^{N\times N}$, then we have

$$[C,D](f_n(x,y;A)t^n) = -f_n(x,y;A)t^n$$

and our claim is justified.
Now, the relation $Dy_n = -(n+1)y_{n+1}$ gives following differential recurrence relation on operator C

$$\left[-y^2tI\frac{\partial}{\partial x} + I\lambda ty\right] f_n(x,y;A)t^n = -(n+1)f_{n+1}(x,y;A)t^{n+1}$$

or

$$\left[-y^2I\frac{\partial}{\partial x} + I\lambda y\right] f_n(x,y;A) = -(n+1)f_{n+1}(x,y;A) \tag{3.14}$$

And again from the relation $Cy_n = y_{n-1}$ we obtain the following differential recurrence relation on operator A

$$\left[\frac{-Ix}{\lambda ty}\frac{\partial}{\partial x} - \frac{I}{\lambda y}\frac{\partial}{\partial t} - \frac{1}{\lambda ty}A\right] f_n(x,y;A)t^n = f_{n-1}(x,y;A)t^{n-1}$$

$$\left[\frac{x}{\lambda y}\frac{\partial}{\partial x} + \frac{nI+A}{\lambda y}\right] f_n(x,y;A) = -f_{n-1}(x,y;A) \tag{3.15}$$

Finally, the relation $DCy_n = -ny_n$ gives

$$\left[-y^2t\frac{\partial}{\partial x} + \lambda ty\right]\left[\frac{-x}{\lambda ty}\frac{\partial}{\partial x} - \frac{1}{\lambda y}\frac{\partial}{\partial t} - \frac{1}{\lambda ty}A\right] f_n(x,y;A)t^n = -nIf_n(x,y;A)t^n$$

equivalently

$$-\frac{xy}{\lambda}\frac{\partial^2}{\partial x^2} - \left[\frac{Ay}{\lambda} + \left\{\frac{y}{\lambda}(n+1) - x\right\} I\frac{\partial}{\partial x} - A\right] f_n(x,y;A) = -nf_n(x,y;A) \tag{3.16}$$

Now, we observe that 2- variables matrix Laguerre polynomial $L_n^{(A,\lambda)}(x,y)$ is the solution of the differential equation (3.16). Further we note that the relation (3.14) and (3.15) are differential recurrence relation satisfied by Laguerre Matrix polynomial $L_n^{(A,\lambda)}(x,y)$.

4 Conclusion

A new approach has been introduced in this paper for studying some important properties of certain matrix special functions viz. recurrence relation, differential recurrence relation and differential equation. The method developed in this paper can also be used to study some other special matrix functions which play vital role in Mathematical Physics, Chemistry and Mechanics.

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