Certain Basic Hypergeometric Series Identities Through $q$-Exponential Operator Technique

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Abstract

In the present work by making use of $q$-exponential operator technique along with the well known Heine’s and Jackson’s transformations, we have obtained certain new basic hypergeometric series identities. Some interesting applications of new results have also been discussed.

1 Introduction

For $|q| < 1$ and $\alpha$ (real or complex), we define the $q$-shifted factorial as

$$(a; q)_n = (1 - a)(1 - aq)\ldots (1 - aq^{n-1}), \quad n = 1, 2, 3\ldots$$

and

$$(a; q)_0 = 1.$$ 

Also

$$(a)_\infty = (a; q)_\infty = \prod_{n=0}^{\infty} (1 - aq^n)$$

and

$$(a)_n = \frac{(a)_\infty}{(aq^n)_\infty}.$$ 

We also employ the notation

$$(\alpha_1, \alpha_2, \ldots, \alpha_k; q) = (\alpha_1; q)(\alpha_2; q)\ldots(\alpha_k; q).$$

Further, recall the definition of basic hypergeometric series $[1]$

$$\phi_r\left( \begin{array}{c} a_1, a_2, \ldots, a_r \\ b_1, b_2, \ldots, b_s \end{array} ; q, z \right) = \sum_{n=0}^{\infty} \frac{(a_1)_n(a_2)_n\ldots(a_r)_n}{(b_1)_n\ldots(b_s)_n} \frac{(-1)^n q^{n(n-1)} z^n}{(1-q) n!} z^n,$$

where $q \neq 0$ when $r > s + 1$. Note that the generalized basic hypergeometric series is defined by

$$\phi_{r+1}\left( \begin{array}{c} a_1, a_2, \ldots, a_r \\ b_1, b_2, \ldots, b_r \end{array} ; q, z \right) = \sum_{n=0}^{\infty} \frac{(a_1)_n(a_2)_n\ldots(a_{r+1})_n}{(b_1)_n\ldots(b_r)_n} \frac{z^n}{(1-q) n!} z^n,$$

where $|z| < 1$ and $|q| < 1$.

The $q$-differential operator or $q$-derivative is defined as

$$D_qf(a) = \frac{f(a) - f(aq)}{a}.$$  (1.1)

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The Leibnitz rule for $D_q$ is the following identity which is a variation of the $q$-binomial theorem [2]

$$D_q^n f(a)g(a) = \sum_{k=0}^{n} q^{k(n-k)} \binom{n}{k} D_q^{k} f(a) D_q^{n-k} g(q^k a),$$  \hspace{1cm} (1.2) 

where $D_q^0$ is an identity and

$$\binom{n}{k} = \frac{(q)_n}{(q)_k (q)_{n-k}}$$

is the $q$-binomial coefficient.

Chen and Liu [4,5] have introduced an exponential operator technique of parameter augmentation for basic hypergeometric series and have established certain $q$-series identities [4]. In particular, they have constructed the following $q$-exponential operator

$$T(bD_q) = \sum_{n=0}^{\infty} \frac{(bD_q)^n}{(q;q)_n}. \hspace{1cm} (1.3)$$

The operator $T(bD_q)$ satisfies the following identities

$$T(bD_q) \frac{1}{(at)_{\infty}} = \frac{1}{(at, bt)_{\infty}}, \hspace{1cm} (1.4)$$

$$T(bD_q) \frac{1}{(as, at)_{\infty}} = \frac{(abst)_{\infty}}{(as, at, bs, bt)_{\infty}}. \hspace{1cm} (1.5)$$

In present communication, we have used the $q$-exponential operator (1.0.3) along with the well known Heine’s transformations [1]

$$\sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(q)_n (c)_n} z^n = \frac{(b)_\infty (az)_\infty}{(c)_\infty (z)_\infty} \sum_{n=0}^{\infty} \frac{(c/b)_n (z)_n}{(q)_n (az)_n} b^n \hspace{1cm} (1.6)$$

$$= \frac{(c/b)_\infty (b)_\infty}{(c)_\infty (z)_\infty} \sum_{n=0}^{\infty} \frac{(abz/c)_n (b)_n}{(q)_n (b)_n} (c/b)^n \hspace{1cm} (1.7)$$

$$= \frac{(abz/c)_\infty}{(z)_\infty} \sum_{n=0}^{\infty} \frac{(c/a)_n (c/b)_n}{(q)_n (c)_n} (abz/c)^n, \hspace{1cm} (1.8)$$

Jackson’s transformation [1]

$$2\phi_1(a, b; c; q, z) = \frac{az}{z} 2\phi_2(a, c/b; c, az; q, bz) \hspace{1cm} (1.9)$$

and the $q$-Pfaff-Saalschutz formula

$$3\phi_2 \left( \begin{array}{cc} a, & b, \\ c, & \end{array} ; \begin{array}{c} q^{-n} \\ q \end{array} \right) = \frac{(c/a, c/b)_n}{(c/c/ab)_n} \hspace{1cm} (1.10)$$

to establish certain new basic hypergeometric series identities.
2 Main Results

Theorem 1. We have

\[
\left(\begin{array}{cccc}
3 \phi_2 \\
a, & b, & d \\
aq, & aq \\
\end{array} \right)_{; q, z} = \frac{(b, az, adq/b, dq)_\infty}{(aq, z, dq/b, adq)_\infty} \times \left(\begin{array}{cccc}
3 \phi_2 \\
z, & aq/b, & dq/b \\
az, & adq/b \\
\end{array} \right)_{; q, b} = \left(\begin{array}{cccc}
2 \phi_2 \\
bz/q, & aq/b, & dq/b \\
q, & adq/b, & az \\
\end{array} \right)_{; q, bz/q} = \left(\begin{array}{cccc}
2 \phi_2 \\
q, & 0, & adq/b \\
aq, & adq/b, & az \\
\end{array} \right)_{; q, q}\]

(2.1)

Proof: Taking \(c = aq\) in (1.0.6), we obtain

\[
\sum_{n=0}^{\infty} \frac{(b)_n(aq^{n+1})_\infty}{(q)_n(aq^n)_\infty} z^n = \frac{(b, az, aq/b)_\infty}{(z, aq)_\infty} \sum_{n=0}^{\infty} \frac{(z)_n}{(q)_n(aq^n)_\infty(a)_\infty} (b)^n.
\]

Applying the operator \(T(dD_q)\) on both sides and using the operator (1.0.5), after some simplification we obtain (2.0.11).

Putting \(c = aq\) in (1.0.8), we obtain

\[
\sum_{n=0}^{\infty} \frac{(b)_n(aq^{n+1})_\infty}{(q)_n(aq^n)_\infty} z^n = \frac{(bz/q, aq/b)_\infty}{(z)_\infty} \sum_{n=0}^{\infty} \frac{(z)_n}{(q)_n(aq^{n+1/b})_\infty(a)_\infty} (bz/q)^n.
\]

Applying the operator \(T(dD_q)\) on both sides and using the operator (1.0.5), we obtain (2.0.12).

Putting \(c = aq\) in (1.0.9), we get

\[
\sum_{n=0}^{\infty} \frac{(b)_n(aq^{n+1})_\infty}{(q)_n(aq^n)_\infty} z^n = \frac{(bz/q)_\infty}{(z, dq/b)_\infty} \sum_{n=0}^{\infty} \frac{(a)_n(0)(aq^{n+1/b})_\infty(a/b)_\infty q^n}{(q)_n(aq^{n+1/b})_\infty(a)_\infty} \frac{1}{(aq^{n+1/b})_\infty}.
\]

Applying the operator \(T(dD_q)\) on both sides and using the operator (1.0.5), we obtain (2.0.13).

Theorem 2. We have

\[
\sum_{n=0}^{\infty} \frac{(b)_n(aq)_n d_n}{(q)_n(aq)_n} z^n = \frac{(bz/q, adq/b)_\infty}{(z, dq/b)_\infty} \sum_{n=0}^{\infty} \frac{(aq/b)_n(adq/b)_n}{(aq)_n(adq/b)_n} (bz/q)^n.
\]

(2.4)

Proof: Taking \(c = aq\) in (1.0.8), we have

\[
\sum_{n=0}^{\infty} \frac{(b)_n z^n}{(q)_n(aq)_n} \frac{1}{(aq^n)_\infty} = \frac{(bz/q)_\infty}{(z)_\infty} \sum_{n=0}^{\infty} \frac{(z)_n}{(aq)_n(aq^{n+1/b})_\infty(a)_\infty} \frac{1}{(aq^{n+1/b})_\infty}.
\]

Applying the operator \(T(dD_q)\) on both sides and then (1.0.4) in L.H.S. and (1.0.5) in R.H.S., we obtain (2.0.14).

Theorem 3. We have

\[
\left(\begin{array}{cccc}
4 \phi_3 \\
a, & b, & q^{-n}, & d \\
c, & ab^{-1}q^{-1-n}, & ade^{-1}q^{-1-n} \\
\end{array} \right)_{; q, q} = \frac{(c/a, c/b, c/d)_n}{(c, c/ab, c/ad)_n}.
\]

(2.5)
Proof: In (1.0.10), using the identity
\[
(c/a)_n = (-c/a)^n q^{n(n-1)/2} \frac{(aq^{1-n}/c)_\infty}{(aq/c)_\infty}
\]
we obtain
\[
\sum_{k=0}^{n} \frac{(b)_k (q^{-n}) k q^k}{(c)_k (abc^{-1} q^{1-n}) k (aq^k, aq^{1-n}/c)_\infty} = (-c/a)^n q^{n(n-1)/2} (c/b)_n
\]
\[
(c/a)_n (aq/c)_\infty.
\]
Applying the operator \( T(dD_q) \) on both sides and using the operator (1.0.5), we obtain (2.0.15).

3 Special Cases

Taking \( b = aq \) in (2.0.14), we get
\[
\sum \frac{(a)_n (d)_n}{(q)_n} z^n = \frac{(az)_\infty (d)_\infty}{(z)_\infty (d/a)_\infty}. \tag{3.1}
\]
If we take \( z = q/a \) and then \( a \to \infty \) in (3.0.16), we obtain the following identity
\[
\sum (-)^n q^{n(n+1)/2} (q^{n+1})_\infty (d)_n = (q^2)_\infty (d)_\infty \tag{3.2}
\]
which for \( d = q \) gives
\[
\sum (-)^n q^{n(n+1)/2} = (q^2)_\infty.
\]
Again, taking \( b \to 1 \) in (2.0.14), we get
\[
\sum_{n=0}^{\infty} \frac{(dq)_n}{(aq)_n} (z/q)^n = \frac{(z)_\infty (dq)_\infty}{(z/q)_\infty (adq)_\infty}
\]
which for \( dq = B \) can be written as
\[
\sum_{n=0}^{\infty} \frac{(B)_n}{(aB)_n} (z/q)^n = \frac{(B)_\infty}{(1 - z/q)(aB)_\infty}. \tag{3.3}
\]
For \( d \to 0 \) in (2.0.14), we obtain the following transformation formula
\[
z \phi_1 \left( \begin{array}{c} a, b \\aq \end{array} ; q, z \right) = \frac{(bz/q)_\infty}{(z)_\infty} \sum_{n=0}^{\infty} \frac{(aq/b)_n}{(aq)_n} (bz/q)^n. \tag{3.4}
\]
In (2.0.11), putting \( d = aq \) and then taking \( b=q \), we obtain
\[
\sum_{n=0}^{\infty} (a)_n (aq^2)_n z^n = \frac{(q, az, a^2 q, a^2 q^2)_\infty}{(aq, z, aq, a^2 q^2)_\infty} \sum_{n=0}^{\infty} (z)_n (a)_n (aq)_n q^n
\]
which for \( z=q \) gives the following identity
\[
\sum_{n=0}^{\infty} (a)_n (aq^2)_n q^n = \frac{(a^2 q, a^2 q^2)_\infty}{(aq, a^2 q^2)_\infty} \sum_{n=0}^{\infty} (a^2 q)_n q^n. \tag{3.5}
\]

References


